

Infinitely many solutions to a fractional nonlinear Schrödinger equation

Liping Wang[†] and Chunyi Zhao^{*}

Department of Mathematics,
Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice,
East China Normal University, Shanghai, 200241, China

March 4, 2014

Abstract

This paper considers the fractional Schrödinger equation

$$(-\Delta)^s u + V(|x|)u - u^p = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N) \quad (0.1)$$

where $0 < s < 1$, $1 < p < \frac{N+2s}{N-2s}$, $V(|x|)$ is a positive potential and $N \geq 2$. We show that if $V(|x|)$ has the following expansion:

$$V(|x|) = V_0 + \frac{a}{|x|^m} + o\left(\frac{1}{|x|^m}\right) \quad \text{as } |x| \rightarrow +\infty,$$

in which the constants are properly assumed, then (0.1) admits infinitely many non-radial solutions, whose energy can be made arbitrarily large. This is the first result for fractional Schrödinger equation. The $s = 1$ case corresponds to the known result in Wei-Yan [28].

Key Words. Fractional Laplacian, fractional Schrödinger equation, Lyapunov-Schmidt

1 Introduction and main results

The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics. The nonlinear fractional Schrödinger equation is as follows:

$$i\psi_t = (-\Delta)^s \psi + \tilde{V}(x)\psi - |\psi|^{p-1}\psi \quad (1.1)$$

where $(-\Delta)^s$ ($0 < s < 1$) denotes the classical fractional Laplacian, \tilde{V} is a bounded potential and $p > 1$.

We are interested in finding *standing wave solutions*, which are solutions of the form $\psi(x, t) = u(x)e^{i\lambda t}$ with the function u real-valued. Let $V(x) = \tilde{V}(x) + \lambda$, then ψ is a solution of (1.1) if and only if u solves the following equation

$$(-\Delta)^s u + V(x)u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N. \quad (1.2)$$

A similar problem to (1.2) is the following fractional scalar field equation

$$(-\Delta)^s u + u = Q(x)u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N. \quad (1.3)$$

It is also absorbing to study the singularly perturbed problem

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N \quad (1.4)$$

or

$$\varepsilon^{2s}(-\Delta)^s u + u = Q(x)u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N \quad (1.5)$$

[†]lpwang@math.ecnu.edu.cn

^{*}Corresponding author. cyzhao@math.ecnu.edu.cn

where $\varepsilon > 0$ is a small parameter. The natural place to look for solutions that decay at infinity is the space $H^{2s}(\mathbb{R}^N)$ of all functions $u \in L^2(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (1 + |\xi|^{4s}) |\hat{u}(\xi)|^2 d\xi < \infty,$$

where $\widehat{\cdot}$ denotes the Fourier transform. The fractional Laplacian $(-\Delta)^s u$ for $u \in H^{2s}(\mathbb{R}^N)$ is defined by

$$(\widehat{-\Delta})^s u(\xi) = |\xi|^{2s} \hat{u}(\xi).$$

For (1.2)–(1.5), an interesting problem is to find solutions with a spike pattern concentrating around some points. As for the standard case $s = 1$ of (1.4) or (1.5), this has been the topic of many works relating the concentration points with critical points of the potential, starting in 1986 from the pioneering work Floer-Weinstein [17]. Later many works show that the number of the critical points of $V(x)$ (or $Q(x)$) (see for example [1, 6, 7, 11, 12, 13, 14, 24, 27]), the type of the critical points of $V(x)$ (or $Q(x)$) (see for example [9, 21, 23, 30]), and the topology of the level set $V(x)$ (or $Q(x)$) (see for example [2, 3, 8, 16]), can effect the number of solutions of (1.4) (or (1.5)). It is now known that when the parameter ε goes to zero, the number of the solutions may tend to infinity. For the $s = 1$ case of (1.2) or (1.3), in 2010 Wei-Yan [28] get a multiplicity result under some symmetry assumption of $V(x)$ near the infinity. Recently we are told that del Pino-Wei-Yao [15] get the similar result with a weaker symmetry assumption on $V(x)$.

As to the fractional case $0 < s < 1$, very few is known. Recently Dávila-del Pino-Wei [10] obtained the first result of spike pattern for the fractional Schrödinger equation (1.4) with $1 < p < \frac{N+2s}{N-2s}$. A natural question is can we get multiplicity result for (1.2) (or (1.3)) with $0 < s < 1$? What is the situation in the fractional case? In this paper we will give an affirmative answer!

This paper is concerned about the following fractional Laplacian problem

$$(-\Delta)^s u + V(|x|)u - |u|^{p-1}u = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N) \quad (1.6)$$

where $0 < s < 1$, $1 < p < \frac{N+2s}{N-2s}$ and $N \geq 2$. We suppose that $V(x)$ satisfies the following assumption.

Assumption \mathcal{V} . V is positive and radially symmetric, i.e. $V(x) = V(|x|) > 0$ and there are constants $a > 0$ and $V_0 > 0$ such that

$$V(|x|) = V_0 + \frac{a}{|x|^m} + o\left(\frac{1}{|x|^m}\right), \quad \text{as } |x| \rightarrow +\infty, \quad (1.7)$$

where

$$\max \left\{ 0, (N+2s) \left[1 - (p-1)N - 2ps + \max \left\{ s, p - \frac{N}{2} \right\} \right] \right\} < m < N + 2s. \quad (1.8)$$

Without loss of generality, we may assume $V_0 = 1$ for the sake of simplicity.

It's easy to see that

$$\left[1 - (p-1)N - 2ps + \max \left\{ s, p - \frac{N}{2} \right\} \right] < 1 \quad \text{for any } p > 1, s \in (0, 1).$$

By direct computations we find that in three dimension case, if

$$1 + \frac{1-s}{3+2s} < p < \frac{3+2s}{3-2s}, \quad \frac{1}{6} < s \leq \frac{1}{2},$$

then we just need that $m \in (0, N + 2s)$.

The aim of this paper is to obtain *infinitely many non-radial positive solutions* to (1.6), whose energy may be arbitrarily large. Our main result in this paper is stated in the following theorem.

Theorem 1.1. *If $V(|x|)$ satisfies the assumption \mathcal{V} , then the problem (1.6) admits infinitely many non-radial positive solutions. Moreover, the energy of these solutions may be arbitrarily large.*

Remark 1.1. *The condition on potential $V(x)$ is more general than that of V in [28] for $s = 1$. The main reason is that in our case we can deduce the exact relationship between the radius and the number of spikes, while in [28], the authors can't solve it exactly using the leading terms of energy.*

We believe that the symmetry on V is technical and then make the following conjecture.

Conjecture 1.1. *Problem (1.6) has infinitely many solutions if there are constants $a > 0, m \in (0, N+2s)$ and $V_0 > 0$, such that*

$$V(x) = V_0 + \frac{a}{|x|^m} + o\left(\frac{1}{|x|^m}\right), \quad \text{as } |x| \rightarrow +\infty.$$

Remark 1.2. *Using the same argument, we can prove that if*

$$Q(|x|) = Q_0 - \frac{a}{|x|^m} + o\left(\frac{1}{|x|^m}\right) \quad \text{as } |x| \rightarrow +\infty,$$

where the constants are similarly assumed, then problem (1.3) has infinitely many positive non-radial solutions.

Before close this introduction, let us outline the main idea in the proof of Theorem 1.1. Our aim is to construct solutions with a large number of bumps near the infinity. Since

$$\lim_{|x| \rightarrow +\infty} V(|x|) = 1,$$

we will use the solution of

$$(-\Delta)^s u + u - |u|^{p-1} u = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N) \quad (1.9)$$

to build up the approximate solution for problem (1.6). It is known (see for instance [20]) the existence of a positive, radial least energy solution $w(x)$, which gives the lowest possible value for the energy

$$J_1(v) = \frac{1}{2} \int_{\mathbb{R}^N} v(-\Delta)^s v + \frac{1}{2} \int_{\mathbb{R}^N} v^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |v|^{p+1}$$

among all nontrivial solutions of (1.9). An important property, which has been proven recently by Frank-Lenzman-Silvestre [20] (see also [4, 19]), is that there exists a radial least energy solution which is non-degenerate, in the sense that the space of solutions of the equation

$$(-\Delta)^s \phi + \phi - p w^{p-1} \phi = 0, \quad \phi \in H^{2s}(\mathbb{R}^N) \quad (1.10)$$

consists exactly of the linear combinations of the translation-generators $\frac{\partial w}{\partial x_j}, j = 1, \dots, N$. Also we have the following behavior for $w(x)$ ([20]):

$$w'(|x|) < 0; \quad w(|x|) = \frac{A}{|x|^{N+2s}} (1 + o(1)), \quad A > 0, \quad \text{as } |x| \rightarrow +\infty. \quad (1.11)$$

Let

$$q_j = \left(r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, \mathbf{0} \right), \quad j = 1, \dots, k,$$

where $\mathbf{0}$ is the zero vector in \mathbb{R}^{N-2} , $r \in \left[\frac{1}{C_0} k^{\frac{N+2s}{N+2s-m}}, C_0 k^{\frac{N+2s}{N+2s-m}} \right]$ for large positive constant C_0 independent of k .

Set $x = (x', x'')$, $x \in \mathbb{R}^2$, $x'' \in \mathbb{R}^{N-2}$. Define

$$H_s = \left\{ u \mid u \in H^{2s}(\mathbb{R}^N), u \text{ is even in } x_h \ (h = 2, \dots, N) \text{ and} \right. \\ \left. u(r \cos \theta, r \sin \theta, x'') = u\left(r \cos\left(\theta + \frac{2\pi j}{k}\right), r \sin\left(\theta + \frac{2\pi j}{k}\right), x''\right), \ j = 1, \dots, k-1 \right\}.$$

Define

$$W(x) = \sum_{j=1}^k w(x - q_j),$$

then Theorem 1.1 is a direct consequence of the following result.

Theorem 1.2. *Suppose $V(|x|)$ satisfies the assumption \mathcal{V} . Then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, Problem (1.6) has a solution u_k of the form*

$$u_k(x) = W(x) + \varphi(x),$$

where $\varphi(x) \in H_s$ and the energy at u_k goes to infinity as k goes to infinity.

Remark 1.3. Note that there is no parameter in the problem (1.6). Using the number of spikes as parameter, we get the **first multiplicity result** for fractional nonlinear Schrödinger equation, **which seems a new phenomenon** for fractional nonlinear Schrödinger equation.

Remark 1.4. Since the approximate solution has polynomial decay, we should deal with every term carefully in the calculus which makes our proof a little bit complicated. By the way, in [28], the approximation has exponential decay.

The paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, the ansatz is established. In Section 4, we deal with the corresponding linearized problem. In Section 5, the nonlinear problem is considered and the proof of Theorem 1.2 is given. Finally some important estimates and the expansion of the energy are stated in Section 6.

Notations. In what follows, the symbol C always denotes a various constant independent of k .

2 Preliminaries

In this section, we get a useful a-priori estimate for a related linear equation.

Let $0 < s < 1$. Various definitions of the fractional Laplacian $(-\Delta)^s \varphi$ of a function φ defined in \mathbb{R}^N are available, depending on its regularity and growth properties, see for example [10]. A useful (local) representation given by Caffarelli and Silvestre [5], is via the following boundary value problem in the half space $\mathbb{R}_+^{N+1} = \{(x, y) \mid x \in \mathbb{R}^N, y > 0\}$:

$$\nabla \cdot (y^{1-2s} \nabla \tilde{\varphi}) = 0 \quad \text{in } \mathbb{R}_+^{N+1}, \quad \tilde{\varphi}(x, 0) = \varphi(x) \quad \text{on } \mathbb{R}^N.$$

Here $\tilde{\varphi}$ is the s -harmonic extension of φ , explicitly given as a convolution integral with the s -Poisson kernel $p_s(x, y)$,

$$\tilde{\varphi}(x, y) = \int_{\mathbb{R}^N} p_s(x - z, y) \varphi(z) dz,$$

where

$$p_s(x, y) = c_{N,s} \frac{y^{4s-1}}{(|x|^2 + |y|^2)^{\frac{N-1+4s}{2}}}$$

and $c_{N,s}$ achieves $\int_{\mathbb{R}^N} p_s(x, y) dx = 1$. Then under suitable regularity, $(-\Delta)^s \varphi$ is the Dirichlet-to-Neumann map for this problem, that is

$$(-\Delta)^s \varphi(x) = \lim_{y \rightarrow 0^+} y^{1-2s} \partial_y \tilde{\varphi}(x, y). \quad (2.1)$$

For $m > 0$ and $g \in L^2(\mathbb{R}^N)$, let us consider now the equation

$$(-\Delta)^s \varphi + m\varphi = g \quad \text{in } \mathbb{R}^N.$$

Then in terms of Fourier transform, for $\varphi \in L^2(\mathbb{R}^N)$, this problem reads

$$(|\xi|^{2s} + m) \hat{\varphi} = \hat{g}$$

and has a unique solution $\varphi \in H^{2s}(\mathbb{R}^N)$ given by the convolution

$$\varphi(x) = T_m(g) := \int_{\mathbb{R}^N} k(x - z) g(z) dz \quad (2.2)$$

where the Fourier transform of k is

$$\hat{k}(\xi) = \frac{1}{|\xi|^{2s} + m}.$$

Then we have the following main properties of the fundamental solution $k(x)$ (see for example [20, 18]): $k(x)$ is radially symmetric and positive, $k \in C^\infty(\mathbb{R}^N \setminus \{0\})$ satisfying

$$\begin{aligned} (i) \quad & |k(x)| + |x| |\nabla k(x)| \leq \frac{C}{|x|^{N-2s}} \quad \text{for all } |x| \leq 1; \\ (ii) \quad & \lim_{|x| \rightarrow \infty} k(x) |x|^{N+2s} = \alpha > 0; \\ (iii) \quad & |x| |\nabla k(x)| \leq \frac{C}{|x|^{N+2s}} \quad \text{for all } |x| \geq 1. \end{aligned}$$

Using (2.1) written in weak form, φ can be characterized by $\varphi(x) = \tilde{\varphi}(x, 0)$ in trace sense, where $\tilde{\varphi} \in H$ is the unique solution of

$$\int_{\mathbb{R}_+^{N+1}} \nabla \tilde{\varphi}(x, y) \cdot \nabla \phi(x, y) y^{1-2s} dx dy + m \int_{\mathbb{R}^N} \varphi(x) \phi(x, 0) dx = \int_{\mathbb{R}^N} g(x) \phi(x, 0) dx, \quad (2.3)$$

for all $\phi \in H$, where H is the Hilbert space of functions $\phi \in H_{\text{loc}}^1(\mathbb{R}_+^{N+1})$ such that

$$\|\phi\|_H^2 := \int_{\mathbb{R}_+^{N+1}} |\nabla \phi(x, y)|^2 y^{1-2s} dx dy + m \int_{\mathbb{R}^N} |\phi(x, 0)|^2 dx < +\infty,$$

or equivalent the closure of the set of all functions in $C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$ under this norm.

For our purpose, we need the following four lemmas, see [10]:

Lemma 2.1. *Let $g \in L^2(\mathbb{R}^N)$. Then the unique solution $\tilde{\varphi} \in H$ of the problem (2.3) is given by the s -harmonic extension of the function $\varphi = T_m(g)$.*

Lemma 2.2. *Let $0 \leq \mu < N + 2s$. Then there exists a positive constant C such that*

$$\|(1 + |x|)^\mu T_m(g)\|_{L^\infty(\mathbb{R}^N)} \leq C \|(1 + |x|)^\mu g\|_{L^\infty(\mathbb{R}^N)}.$$

Lemma 2.3. *Assume that $g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then the following holds: if $\varphi = T_m(g)$ then there is a $C > 0$ such that*

$$\sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\beta} \leq C \|g\|_{L^\infty(\mathbb{R}^N)} \quad (2.4)$$

where $\beta = \min\{1, 2s\}$.

Lemma 2.4. *Let $\varphi \in H^{2s}$ be the solution of*

$$(-\Delta)^s \varphi + W(x) \varphi = g \quad \text{in } \mathbb{R}^N \quad (2.5)$$

with bounded potential W . If $\inf_{x \in \mathbb{R}^N} W(x) =: m > 0$, $g \geq 0$. Then $\varphi \geq 0$ in \mathbb{R}^N .

Using these lemmas, we obtain an a-priori estimate for any solution $\varphi \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ of (2.5).

Lemma 2.5. *Let W be a continuous function, and assume that for k points q_1, \dots, q_k , there is an $R > 0$ and $B = \cup_{j=1}^k B_R(q_j)$ such that*

$$\inf_{x \in \mathbb{R}^N \setminus B} W(x) =: m > 0.$$

Then given any number $\frac{N}{2} < \mu < N + 2s$, there exists a uniform positive constant $C = C(\mu, R)$ independent of k such that for any $\varphi \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and g satisfying (2.5) with

$$\|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)} < +\infty, \quad \text{where } \rho(x) = \sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^\mu},$$

we have the validity of the estimate

$$\|\rho^{-1} \varphi\|_{L^\infty(\mathbb{R}^N)} \leq C (\|\varphi\|_{L^\infty(B)} + \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)}).$$

Proof. We rewrite (2.5) as

$$(-\Delta)^s \varphi + \widetilde{W} \varphi = \tilde{g},$$

where $\tilde{g} = (m - W)\chi_B \varphi + g$, $\widetilde{W} = m\chi_B + W(1 - \chi_B)$ and χ_B is the characteristic function on B . By careful calculation, it is deduced that

$$|\tilde{g}(x)| \leq C \|\varphi\|_{L^\infty(B)} + \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)} \rho \leq M \rho$$

where

$$\begin{aligned} M &= C \|\varphi\|_{L^\infty(B)} \sup_{x \in B} \left(\sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^\mu} \right)^{-1} + \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)} \\ &\leq C \|\varphi\|_{L^\infty(B)} \max_{1 \leq j \leq k} \sup_{x \in B_R(q_j)} \left(\frac{1}{(1 + |x - q_j|)^\mu} \right)^{-1} + \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)} \\ &\leq C \|\varphi\|_{L^\infty(B)} (1 + R^\mu) + \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)} \\ &\leq C(\mu, R) (\|\varphi\|_{L^\infty(B)} + \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)}). \end{aligned}$$

From Lemma 2.2 with $0 < \mu < N + 2s$, the positive solution φ_0 to the problem

$$(-\Delta)^s \varphi_0 + m\varphi_0 = \frac{1}{(1 + |x|)^\mu}$$

satisfies $\varphi_0 = O(|x|^{-\mu})$ as $|x| \rightarrow +\infty$. Since $\inf_{x \in \mathbb{R}^N} \widetilde{W}(x) \geq m$ obviously, we have

$$\left((- \Delta)^s + \widetilde{W}\right) \bar{\varphi} \geq M \sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^\mu}$$

where $\bar{\varphi}(x) = M \sum_{j=1}^k \varphi_0(x - q_j)$. Setting $\psi = \varphi - \bar{\varphi}$, one find that

$$(-\Delta)^s \psi + \widetilde{W}\psi = \bar{g} \leq 0.$$

Using Lemma 2.4 we get that $\varphi \leq \bar{\varphi}$. Arguing similarly for $-\varphi$, we get that $|\varphi| \leq \bar{\varphi}$. Then it holds that

$$\begin{aligned} \|\rho^{-1}\varphi\|_{L^\infty(\mathbb{R}^N)} &\leq \|\rho^{-1}\bar{\varphi}\|_{L^\infty(\mathbb{R}^N)} = M \left\| \left(\sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^\mu} \right)^{-1} \sum_{i=1}^k \varphi_0(x - q_i) \right\|_{L^\infty(\mathbb{R}^N)} \\ &\leq CM \left\| \left(\sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^\mu} \right)^{-1} \sum_{i=1}^k \frac{1}{(1 + |x - q_i|)^\mu} \right\|_{L^\infty(\mathbb{R}^N)} \\ &\leq CM. \end{aligned}$$

The desired estimate follows right now. \square

Examining the above proof, we can deduce the following immediately.

Corollary 2.1. *Let $\rho(x)$ be defined as in the previous lemma. Assume that $\varphi \in H^{2s}(\mathbb{R}^N)$ satisfies the problem (2.5) and that*

$$\inf_{x \in \mathbb{R}^N} W(x) =: m > 0.$$

Then we have that $\varphi \in L^\infty(\mathbb{R}^N)$ and it satisfies

$$\|\rho^{-1}\varphi\|_{L^\infty(\mathbb{R}^N)} \leq C\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)} \quad (2.6)$$

where $C = C(\mu)$ independent of k .

Remark 2.1. *We build these results for any $\frac{N}{2} < \mu < N + 2s$, but for our purpose, from now on we choose*

$$\mu = \frac{N}{2} - \frac{m}{N + 2s} + 1 + \sigma \in \left(\frac{N}{2}, N + 2s\right).$$

Here $\sigma > 0$ is small enough.

Due to the symmetry, we define Ω_j as follows

$$\Omega_j = \left\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{y'}{|y'|}, \frac{q_j}{|q_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\},$$

and introduce the following estimate for later use. For any $\beta \geq \frac{N+2s-m}{N+2s}$ and fixed ℓ , as $k \rightarrow \infty$, it holds that

$$\sum_{i \neq \ell} \frac{1}{|q_i - q_\ell|^\beta} = \frac{1}{2^\beta} \sum_{i \neq \ell} \frac{1}{r^\beta \sin^\beta \frac{|i-\ell|\pi}{k}} \leq \frac{Ck^\beta}{r^\beta} \sum_{i=1}^k \frac{1}{i^\beta} \leq \begin{cases} \frac{Ck^\beta}{r^\beta} = O(r^{-\frac{m\beta}{N+2s}}) & \beta > 1, \\ \frac{Ck^\beta \ln k}{r^\beta} = O(r^{-\frac{m\beta}{N+2s}} \ln r) & \beta = 1, \\ \frac{Ck}{r^\beta} = O(r^{-(\beta - \frac{N+2s-m}{N+2s})}) & \beta < 1. \end{cases}$$

Remark 2.2. *It holds that*

$$\rho(x) \leq C + C \sum_{j=2}^k \frac{1}{|q_1 - q_j|^{\frac{N}{2} - \frac{m}{N+2s} + 1 + \sigma}} \leq C + C \left(\frac{k}{r} \right)^{\frac{N}{2} - \frac{m}{N+2s} + 1 + \sigma} \leq C.$$

according to Lemma 6.1. Also we easily have

$$\int_{\Omega_1} \rho^2 \leq \int_{\Omega_1} \left(\frac{1}{(1 + |x - q_1|)^{\frac{N}{2} - \frac{m}{N+2s} + 1 + \sigma}} + \frac{1}{(1 + |x - q_1|)^{\frac{N}{2} + \sigma}} \sum_{j=2}^k \frac{1}{|q_1 - q_j|^{1 - \frac{m}{N+2s}}} \right)^2 dx \leq C.$$

In what follows, we use $\|f\|_*$ to mean $\|\rho^{-1}f\|_{L^\infty(\mathbb{R}^N)}$ for convenience, i.e.

$$\|f\|_* = \|\rho^{-1}f\|_{L^\infty(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \left(\sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^{\frac{N}{2} - \frac{m}{N+2s} + 1 + \sigma}} \right)^{-1} f(x).$$

A useful fact is that if $f, g \in L^2(\mathbb{R}^N)$ and $F = T_m(f), G = T_m(g)$, then the following holds

$$\int_{\mathbb{R}^N} G(-\Delta)^s F - \int_{\mathbb{R}^N} F(-\Delta)^s G = - \int_{\mathbb{R}^N} T_m(f)g + \int_{\mathbb{R}^N} fT_m(g) = 0$$

since the kernel k is radially symmetric.

3 Ansatz

In this section, we set up the approximation solution and estimate the corresponding error term. By a solution of the problem

$$(-\Delta)^s u + Vu - u^p = 0 \quad \text{in } \mathbb{R}^N,$$

we mean a $u \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that the above equation is satisfied. Let us observe that it suffices to solve

$$(-\Delta)^s u + Vu - u_+^p = 0 \quad \text{in } \mathbb{R}^N \quad (3.1)$$

where $u_+ = \max\{u, 0\}$ with the help of Lemma 2.4.

We look for a solution u of the form

$$u = W + \varphi, \quad W = \sum_{j=1}^k W_j, \quad W_j = w(x - q_j)$$

where $\varphi \in H_s$ is a small function, disappearing as $k \rightarrow +\infty$. In terms of φ , the equation (3.1) becomes

$$(-\Delta)^s \varphi(x) + V(|x|)\varphi(x) - pW^{p-1}\varphi(x) = E + N(\varphi) \quad \text{in } \mathbb{R}^N, \quad (3.2)$$

where

$$\begin{aligned} N(\varphi) &= (W + \varphi)_+^p - W^p - pW^{p-1}\varphi, \\ E &= \sum_{j=1}^k (1 - V(|x|)) W_j + \left(\sum_{j=1}^k W_j \right)^p - \sum_{j=1}^k W_j^p. \end{aligned}$$

Rather than solving the problem (3.2) directly, we shall first solve a projected version of it, precisely,

$$\begin{cases} (-\Delta)^s \varphi(x) + V(|x|)\varphi(x) - pW^{p-1}\varphi(x) = E + N(\varphi) + c \sum_{j=1}^k Z_j & \text{in } \mathbb{R}^N, \\ \varphi \in H_s, \\ \int_{\mathbb{R}^N} Z_j \varphi = 0, & j = 1, \dots, k, \end{cases} \quad (3.3)$$

for some pair (φ, c) where $\varphi \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, c is a constant,

$$Z_j = \frac{\partial W_j}{\partial r} \quad \text{for } j = 1, \dots, k, \quad \text{and} \quad |Z_j| \leq \frac{C}{(1 + |x - q_j|)^{N+2s}} \text{ obviously.}$$

After the problem (3.3) solved, a variational process will carry out to find a suitable r and then make the constant c in (3.3) be zero, i.e. we solve the problem (3.2).

At the end of this section, we give the estimate of E .

Lemma 3.1. *It holds that*

$$\|E\|_* \leq C \left(\frac{k}{r} \right)^{\min\{N+2s, (N+2s)p-\mu\}} + \frac{C}{r^{N+2s-\mu}} + \frac{C}{r^m} = o\left(\frac{1}{r^{m/2}}\right). \quad (3.4)$$

Proof. By symmetry, we just assume that $x \in \Omega_1$ in the following proof. Obviously we know

$$|x - q_j| \geq |x - q_1| \quad \text{for } j = 2, \dots, k.$$

If $|x| \geq |q_1|/2 = r/2$, then

$$V(|x|) - 1 = O\left(\frac{1}{|x|^m}\right) = O\left(\frac{1}{r^m}\right)$$

and in this region

$$\begin{aligned} \left| \rho^{-1} \sum_{j=1}^k (1 - V(|x|)) W_j \right| &\leq \frac{C}{r^m} \left(\sum_{i=1}^k \frac{1}{(1 + |x - q_i|)^\mu} \right)^{-1} \sum_{j=1}^k W_j \\ &\leq \frac{C}{r^m} \left(\sum_{i=1}^k \frac{1}{(1 + |x - q_i|)^\mu} \right)^{-1} \sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^{N+2s}} \\ &\leq \frac{C}{r^m} \left(\sum_{i=1}^k \frac{1}{(1 + |x - q_i|)^\mu} \right)^{-1} \sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^\mu} \\ &\leq \frac{C}{r^m}. \end{aligned}$$

While for $|x| \leq r/2$, then

$$|x - q_1| \geq |q_1| - |x| \geq \frac{r}{2} \quad \text{and} \quad |x - q_j| \geq \frac{r}{2} \quad \text{for } j = 2, \dots, k.$$

Hence

$$\begin{aligned} \left| \rho^{-1} \sum_{j=1}^k (1 - V(|x|)) W_j \right| &\leq C \rho^{-1} \sum_{j=1}^k W_j \leq C \rho^{-1} \sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^{N+2s}} \\ &\leq C \rho^{-1} \sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^\mu} \cdot \frac{1}{(1 + |x - q_j|)^{N+2s-\mu}} \\ &\leq C \left(\sum_{i=1}^k \frac{1}{(1 + |x - q_i|)^\mu} \right)^{-1} \sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^\mu} \cdot \frac{1}{r^{N+2s-\mu}} \\ &\leq \frac{C}{r^{N+2s-\mu}}. \end{aligned}$$

For the other part in E , we observe that

$$\left| \left(\sum_{j=1}^k W_j \right)^p - \sum_{j=1}^k W_j^p \right| \leq C W_1^{p-1} \sum_{j=2}^k W_j + C \sum_{j=2}^k W_j^p + C \left(\sum_{j=2}^k W_j \right)^p.$$

In the case of $\mu \leq (N + 2s)(p - 1)$,

$$\begin{aligned}
\rho^{-1} W_1^{p-1} \sum_{j=2}^k W_j &\leq C \rho^{-1} \frac{1}{(1 + |x - q_1|)^{(N+2s)(p-1)}} \sum_{j=2}^k \frac{1}{(1 + |x - q_j|)^{N+2s}} \\
&\leq C (1 + |x - q_1|)^\mu \frac{1}{(1 + |x - q_1|)^{(N+2s)(p-1)}} \sum_{j=2}^k \frac{1}{(1 + |x - q_j|)^{N+2s}} \\
&\leq C \sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} \leq C \left(\frac{k}{r} \right)^{N+2s};
\end{aligned}$$

Otherwise if $\mu > (N + 2s)(p - 1)$, then

$$\begin{aligned}
\rho^{-1} W_1^{p-1} \sum_{j=2}^k W_j &\leq C \rho^{-1} \frac{1}{(1 + |x - q_1|)^{(N+2s)(p-1)}} \sum_{j=2}^k \frac{1}{(1 + |x - q_j|)^{N+2s}} \\
&\leq C \rho^{-1} \frac{1}{(1 + |x - q_1|)^\mu} \sum_{j=2}^k \frac{1}{(1 + |x - q_j|)^{N+2s-\mu+(N+2s)(p-1)}} \\
&\leq C \sum_{j=2}^k \frac{1}{|q_j - q_1|^{(N+2s)p-\mu}} \leq C \left(\frac{k}{r} \right)^{(N+2s)p-\mu},
\end{aligned}$$

where we used Lemma 6.1. It is easy to deduce that

$$\begin{aligned}
\rho^{-1} \sum_{j=2}^k W_j^p &\leq C \rho^{-1} \sum_{j=2}^k \frac{1}{(1 + |x - q_j|)^{(N+2s)p-\mu}} \frac{1}{(1 + |x - q_1|)^\mu} \\
&\leq C \sum_{j=2}^k \frac{1}{|q_j - q_1|^{(N+2s)p-\mu}} \leq C \left(\frac{k}{r} \right)^{(N+2s)p-\mu}
\end{aligned}$$

and

$$\begin{aligned}
\rho^{-1} \left(\sum_{j=2}^k W_j \right)^p &\leq C \rho^{-1} \left(\sum_{j=2}^k \frac{1}{(1 + |x - q_j|)^{N+2s-\frac{\mu}{p}}} \frac{1}{(1 + |x - q_1|)^{\frac{\mu}{p}}} \right)^p \\
&\leq C \left(\sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s-\frac{\mu}{p}}} \right)^p \leq C \left(\frac{k}{r} \right)^{(N+2s)p-\mu}.
\end{aligned}$$

The condition (1.8) of m leads obviously to $N + 2s - \frac{\mu}{p} > 1$, $N + 2s - \mu > \frac{m}{2}$ and $(N + 2s)p - \mu > \frac{N+2s}{2}$. Thus we get the desired result by combining these above estimates. \square

4 Linearized theory

This section is devoted to solve a projected linear problem.

We consider the linear problem of finding $\varphi \in H^{2s}(\mathbb{R}^N)$ such that for certain constant c , we have

$$\begin{cases} (-\Delta)^s \varphi + V(|x|)\varphi - pW^{p-1}\varphi = g + c \sum_{j=1}^k Z_j & \text{in } \mathbb{R}^N, \\ \varphi \in H_s, \\ \int_{\mathbb{R}^N} Z_j \varphi = 0, & j = 1, \dots, k. \end{cases} \quad (4.1)$$

The constant c is uniquely determined in terms of φ and g when k is sufficient large from the equation

$$\begin{aligned}
c \int_{\mathbb{R}^N} \sum_{j=1}^k Z_j Z_1 &= \int_{\mathbb{R}^N} [(-\Delta)^s \varphi + V(|x|)\varphi - pW^{p-1}\varphi] Z_1 - \int_{\mathbb{R}^N} g Z_1 \\
&= \int_{\mathbb{R}^N} [(-\Delta)^s Z_1 + V(|x|)Z_1 - pW^{p-1}Z_1] \varphi + O(\|g\|_*) \int_{\mathbb{R}^N} \rho |Z_1| \\
&= \int_{\mathbb{R}^N} [(V-1) + p(W_1^{p-1} - W^{p-1})] Z_1 \varphi + O(\|g\|_*),
\end{aligned} \tag{4.2}$$

where we use Lemma 6.1 to obtain

$$\int_{\mathbb{R}^N} \rho |Z_1| \leq C \left(1 + \sum_{j=2}^k \frac{1}{|q_1 - q_j|^\mu} \right) \int_{\mathbb{R}^N} \frac{1}{(1+|x|)^{N+2s}} dx \leq C.$$

By direct calculation, it is easy to see that

$$\int_{\mathbb{R}^N} Z_1^2 = \int_{\mathbb{R}^N} \left(\frac{\partial w(x - q_1)}{\partial r} \right)^2 dx = \int_{\mathbb{R}^N} \left(\frac{\partial w(x - q_1)}{\partial x_1} \right)^2 dx = \frac{1}{N} \int_{\mathbb{R}^N} (w'(|x|))^2 dx, \tag{4.3}$$

and

$$\begin{aligned}
\sum_{j=2}^k \int_{\mathbb{R}^N} Z_j Z_1 &= \sum_{j=2}^k \int_{\mathbb{R}^N} w'(|x - q_1|) \frac{x - q_1}{|x - q_1|} \cdot \left(-\frac{q_1}{r}\right) w'(|x - q_j|) \frac{x - q_j}{|x - q_j|} \cdot \left(-\frac{q_j}{r}\right) dx \\
&= \sum_{j=2}^k \int_{\mathbb{R}^N} w'(|x|) \frac{x^1}{|x|} w'(|x + q_1 - q_j|) \frac{x + q_1 - q_j}{|x + q_1 - q_j|} \cdot \frac{q_j}{r} dx \\
&= \sum_{j=2}^k \left(\int_{\{|x| \leq \frac{1}{2}|q_1 - q_j|\}} + \int_{\{|x| \geq \frac{1}{2}|q_1 - q_j|\}} \right) w'(|x|) \frac{x^1}{|x|} w'(|x + q_1 - q_j|) \frac{x + q_1 - q_j}{|x + q_1 - q_j|} \cdot \frac{q_j}{r} dx \\
&\leq C \sum_{j=2}^k \frac{1}{|q_1 - q_j|^{N+2s}} \int_{\mathbb{R}^N} |w'(|x|)| dx \leq C \sum_{j=2}^k \frac{1}{|q_1 - q_j|^{N+2s}} = O\left(\left(\frac{k}{r}\right)^{N+2s}\right),
\end{aligned} \tag{4.4}$$

where $x = (x^1, \dots, x^N)$. It implies that $\{Z_j\}_{j=1}^k$ is approximately orthogonal provided k large enough because of the symmetry.

As to the first term in the right hand side of (4.2), we do the following analysis.

$$\begin{aligned}
\left| \int_{\Omega_1} (V(|x|) - 1) Z_1 \varphi \right| &\leq C \|\varphi\|_* \int_{\Omega_1} |V(|x|) - 1| \rho \frac{1}{(1+|x - q_1|)^{N+2s}} dx \\
&\leq C \|\varphi\|_* \left(\int_{\{x \in \Omega_1, |x| \geq |q_1|/2\}} + \int_{\{x \in \Omega_1, |x| \leq |q_1|/2\}} \right) |V(|x|) - 1| \rho \frac{1}{(1+|x - q_1|)^{N+2s}} dx \\
&\leq C \|\varphi\|_* \int_{\{x \in \Omega_1, |x| \geq |q_1|/2\}} \frac{1}{|x|^m} \frac{1}{(1+|x - q_1|)^{N+2s}} \sum_{j=1}^k \frac{1}{(1+|x - q_j|)^\mu} dx \\
&\quad + C \|\varphi\|_* \int_{\{x \in \Omega_1, |x| \leq |q_1|/2\}} \frac{1}{(1+|x - q_1|)^{N+2s}} \sum_{j=1}^k \frac{1}{(1+|x - q_j|)^\mu} dx \\
&\leq C \|\varphi\|_* \frac{1}{r^m} \int_{\{x \in \Omega_1, |x| \geq |q_1|/2\}} \frac{1}{(1+|x - q_1|)^{N+2s}} \left(\frac{1}{(1+|x - q_1|)^\mu} + \sum_{j=2}^k \frac{1}{|q_1 - q_j|^\mu} \right) dx \\
&\quad + C \|\varphi\|_* \int_{\{x \in \Omega_1, |x| \leq |q_1|/2\}} \frac{1}{r^{N+2s}} \left(\frac{1}{(1+|x - q_1|)^\mu} + \sum_{j=2}^k \frac{1}{|q_1 - q_j|^\mu} \right) dx \\
&\leq C \|\varphi\|_* \left(\frac{1}{r^m} + \frac{1}{r^m} \left(\frac{k}{r}\right)^\mu + \frac{1}{r^{\mu+2s}} + \frac{1}{r^{2s}} \left(\frac{k}{r}\right)^\mu \right) = o(\|\varphi\|_*), \quad \text{as } k \rightarrow +\infty.
\end{aligned} \tag{4.5}$$

In addition, note that, for any $j \neq 1$, $\ell \neq 1$ and $j \neq \ell$,

$$\begin{aligned} & \int_{\Omega_\ell} \frac{dx}{(1+|x-q_1|)^{N+2s}(1+|x-q_j|)^\mu} \\ & \leq \frac{C}{|q_j-q_1|^{1-\frac{m}{N+2s}}} \int_{\Omega_\ell} \left[\frac{1}{(1+|x-q_1|)^{\frac{3}{2}N+2s+\sigma}} + \frac{1}{(1+|x-q_j|)^{\frac{3}{2}N+2s+\sigma}} \right] dx \\ & \leq \frac{C}{|q_j-q_1|^{1-\frac{m}{N+2s}}} \left[\frac{1}{|q_\ell-q_1|^{\frac{N}{2}+2s+\sigma}} + \frac{1}{|q_\ell-q_j|^{\frac{N}{2}+2s+\sigma}} \right], \end{aligned}$$

where Lemma 6.3 is used in the first inequality. It is checked that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N \setminus \Omega_1} (V-1)Z_1\varphi \right| \leq C\|\varphi\|_* \sum_{\ell=2}^k \int_{\Omega_\ell} |V(|x|)-1| \frac{\rho(x)}{(1+|x-q_1|)^{N+2s}} dx \\ & \leq C\|\varphi\|_* \sum_{\ell=2}^k \int_{\Omega_\ell} \frac{1}{(1+|x-q_1|)^{N+2s}} \left(\frac{1}{(1+|x-q_1|)^\mu} + \frac{1}{(1+|x-q_\ell|)^\mu} + \sum_{\substack{j=2 \\ j \neq \ell}}^k \frac{1}{(1+|x-q_j|)^\mu} \right) dx \\ & \leq C\|\varphi\|_* \left(\sum_{\ell=2}^k \frac{1}{|q_\ell-q_1|^{\mu+2s}} + \sum_{\ell=2}^k \frac{1}{|q_\ell-q_1|^{\mu+2s-\sigma}} + \sum_{\substack{\ell,j=2 \\ j \neq \ell}}^k \frac{C}{|q_j-q_1|^{1-\frac{m}{N+2s}}} \frac{1}{|q_\ell-q_1|^{\frac{N}{2}+2s+\sigma}} \right) \\ & \quad + C\|\varphi\|_* \sum_{\substack{\ell,j=2 \\ j \neq \ell}}^k \frac{C}{|q_j-q_1|^{1-\frac{m}{N+2s}}} \frac{1}{|q_\ell-q_j|^{\frac{N}{2}+2s+\sigma}} \\ & \leq C\|\varphi\|_* \left(\frac{k}{r} \right)^{\frac{N}{2}+2s}. \end{aligned} \tag{4.6}$$

Thus from (4.5) and (4.6) we get that

$$\int_{\mathbb{R}^N} (V-1)Z_1\varphi = o(\|\varphi\|_*).$$

When $1 < p \leq 2$, it holds that

$$\begin{aligned} & \left| \int_{\Omega_1} (W_1^{p-1} - W^{p-1})Z_1\varphi dx \right| \leq C\|\varphi\|_* \int_{\Omega_1} \left(\sum_{j=2}^k W_j \right)^{p-1} \rho|Z_1| dx \\ & \leq C\|\varphi\|_* \left(\sum_{j=2}^k \frac{1}{|q_1-q_j|^{N+2s}} \right)^{p-1} \int_{\Omega_1} \left[\frac{1}{(1+|x-q_1|)^\mu} + \sum_{j=2}^k \frac{1}{|q_1-q_j|^\mu} \right] \frac{1}{(1+|x-q_1|)^{N+2s}} dx \\ & \leq C\|\varphi\|_* \left(\frac{k}{r} \right)^{(N+2s)(p-1)}, \end{aligned}$$

and, similar to (4.6),

$$\begin{aligned} & \left| \int_{\mathbb{R}^N \setminus \Omega_1} (W_1^{p-1} - W^{p-1})Z_1\varphi \right| dx \leq C\|\varphi\|_* \sum_{\ell=2}^k \int_{\Omega_\ell} \left(\sum_{j=2}^k W_j \right)^{p-1} \rho|Z_1| dx \\ & \leq C\|\varphi\|_* \sum_{\ell=2}^k \int_{\Omega_\ell} \left[\frac{1}{(1+|x-q_\ell|)^{(N+2s)(p-1)}} + \left(\frac{k}{r} \right)^{(N+2s)(p-1)} \right] \\ & \quad \cdot \left[\frac{1}{(1+|x-q_1|)^\mu} + \frac{1}{(1+|x-q_\ell|)^\mu} + \sum_{\substack{j=2 \\ j \neq \ell}}^k \frac{1}{(1+|x-q_j|)^\mu} \right] \frac{dx}{(1+|x-q_1|)^{N+2s}} \end{aligned}$$

$$\leq C\|\varphi\|_* \left(\frac{k}{r}\right)^{\frac{N}{2}+2s}.$$

Thus we obtain, from the above two estimates, that

$$\left| \int_{\mathbb{R}^N} (W_1^{p-1} - W^{p-1}) Z_1 \varphi dx \right| \leq C\|\varphi\|_* \left(\frac{k}{r}\right)^{\min\{(N+2s)(p-1), \frac{N}{2}+2s\}}.$$

For the case $p > 2$, with Lemma 6.2,

$$\begin{aligned} \left| \int_{\Omega_1} (W_1^{p-1} - W^{p-1}) Z_1 \varphi dx \right| &\leq C\|\varphi\|_* \int_{\Omega_1} \left(W_1^{p-2} \sum_{j=2}^k W_j + \left(\sum_{j=2}^k W_j \right)^{p-1} \right) \rho |Z_1| dx \\ &\leq C\|\varphi\|_* \left[\sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} + \left(\sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} \right)^{p-1} \right] \leq C\|\varphi\|_* \left(\frac{k}{r}\right)^{N+2s}, \end{aligned} \quad (4.7)$$

and, also similar to (4.6),

$$\begin{aligned} \left| \int_{\mathbb{R}^N \setminus \Omega_1} (W_1^{p-1} - W^{p-1}) Z_1 \varphi \right| &\leq C\|\varphi\|_* \sum_{\ell=2}^k \int_{\Omega_\ell} \left(W_1^{p-2} \sum_{j=2}^k W_j + \left(\sum_{j=2}^k W_j \right)^{p-1} \right) \rho |Z_1| \\ &\leq C\|\varphi\|_* \left(\frac{k}{r}\right)^{\frac{N}{2}+2s}, \end{aligned} \quad (4.8)$$

on account that, in Ω_ℓ ,

$$\begin{aligned} W_1^{p-2} \sum_{j=2}^k W_j &\leq \frac{C}{|q_\ell - q_1|^{(N+2s)(p-2)}} \left[\frac{1}{(1 + |x - q_\ell|)^{N+2s}} + \sum_{j=2, j \neq \ell}^k \frac{1}{|q_j - q_\ell|^{N+2s}} \right], \\ \left(\sum_{j=2}^k W_j \right)^{p-1} &\leq \frac{C}{(1 + |x - q_\ell|)^{(N+2s)(p-1)}} + C \left(\sum_{j=2, j \neq \ell}^k \frac{1}{|q_j - q_\ell|^{N+2s}} \right)^{p-1}. \end{aligned}$$

So it is concluded from (4.7) and (4.8) that

$$\left| \int_{\mathbb{R}^N} (W_1^{p-1} - W^{p-1}) Z_1 \varphi dx \right| \leq C\|\varphi\|_* \left(\frac{k}{r}\right)^{\frac{N}{2}+2s}.$$

Combining the above inequalities leads to the following lemma right now.

Lemma 4.1. *If (φ, c) solves the problem (4.1), then*

$$c = o(\|\varphi\|_*) + O(\|g\|_*).$$

In the rest of this section we shall build a solution to the problem (4.1).

Proposition 4.1. *Given k large enough, there exists a solution $\varphi = T(g)$ to (4.1) which defines a linear operator of g , provided that $\|g\|_* < +\infty$. Moreover,*

$$\|\varphi\|_* \leq C\|g\|_* \quad \text{and} \quad c \leq C\|g\|_*,$$

where the positive constant C is independent of k .

The key difference of the proof between this proposition and Proposition 4.1 in [10] is that now we should build an a priori estimate which is independent of k , see the coming Lemma 4.2. Once we get such estimate, the remaining is just the same as that in [10].

Lemma 4.2. *Under the assumptions of Proposition 4.1, there exists a positive constant C independent of k such that for any solution φ with $\|\varphi\|_* < +\infty$, we have the following an a priori estimate*

$$\|\varphi\|_* \leq C\|g\|_*.$$

Proof. We argue by contradiction. Suppose that there are $g_k, r_k \in \left[\frac{1}{C_0} k^{\frac{N+2s}{N+2s-m}}, C_0 k^{\frac{N+2s}{N+2s-m}} \right]$ and φ_k solving (4.1) for $g = g_k, r = r_k$ with $\|g_k\|_* \rightarrow 0$ and $\|\varphi_k\|_* \geq C' > 0$. We may assume that $\|\varphi_k\|_* = 1$. For simplicity, we drop the subscript k .

From the conditions of potential V , obviously $\inf_{\mathbb{R}^N} V > 0$. On the other hand, in the equation of φ ,

$$(-\Delta)^s \varphi + (V - pW^{p-1})\varphi = g + c \sum_{j=1}^k Z_j,$$

we find that

$$\begin{aligned} V(x) - pW^{p-1}(x) &\geq V(x) - C \left(\frac{1}{(1 + |x - q_1|)^{N+2s}} + \sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} \right)^{p-1} \\ &\geq V(x) - C \left(\frac{1}{(1 + |x - q_1|)^{N+2s}} + \left(\frac{k}{r}\right)^{N+2s} \right)^{p-1} \geq \frac{1}{2} V(x) \end{aligned}$$

for any $x \in \Omega_1 \setminus B_R(q_1)$, which leads to

$$\inf_{\mathbb{R}^N \setminus \bigcup_{j=1}^k B_R(q_j)} (V(x) - pW^{p-1}(x)) \geq \frac{1}{2} \inf_{\mathbb{R}^N} V(x) > 0.$$

Accordingly, by Lemma 2.5 and Lemma 4.1, it holds that

$$\|\varphi\|_* \leq C \left(\|\varphi\|_{L^\infty(\bigcup_{j=1}^k B_R(q_j))} + \|g\|_* + |c| \left\| \sum_{j=1}^k Z_j \right\|_* \right) \leq C \|\varphi\|_{L^\infty(\bigcup_{j=1}^k B_R(q_j))} + o(1),$$

from which we may assume that, up to a subsequence,

$$\|\varphi\|_{L^\infty(B_R(q_1))} \geq \gamma > 0. \quad (4.9)$$

Let us set $\tilde{\varphi}(x) = \varphi(x + q_1)$, then $\tilde{\varphi}$ satisfies

$$(-\Delta)^s \tilde{\varphi} + V(|x + q_1|)\tilde{\varphi} - pw^{p-1}(x)\tilde{\varphi} = \tilde{g} \quad (4.10)$$

where

$$\begin{aligned} \tilde{g}(x) &= g(x + q_1) + c \left(Z_1(x + q_1) + \sum_{j=2}^k Z_j(x + q_1) \right) \\ &\quad + p \left[\left(w(x) + \sum_{j=2}^k w(x + q_1 - q_j) \right)^{p-1} - w^{p-1}(x) \right] \tilde{\varphi}. \end{aligned} \quad (4.11)$$

For any point x in an arbitrarily compact set of \mathbb{R}^N , we have, from Remark 2.2, that

$$|g(x + q_1)| \leq \|g\|_* \rho(x + q_1) \leq C \|g\|_* = o(1).$$

It is easy to see that $V(x + q_1) \rightarrow 1$,

$$c = o(\|\varphi\|_*) + O(\|g\|_*) \rightarrow 0$$

and

$$\begin{aligned} \left| Z_1(x + q_1) + \sum_{j=2}^k Z_j(x + q_1) \right| &\leq \frac{C}{(1 + |x + q_1|)^{N+2s}} + C \sum_{j=2}^k \frac{1}{|x + q_1 - q_j|^{N+2s}} \\ &\leq C + C \sum_{j=2}^k \frac{1}{|q_1 - q_j|^{N+2s}} \leq C. \end{aligned}$$

For the last term in (4.11), as $1 < p \leq 2$,

$$\begin{aligned}
& \left| \left[\left(w(x) + \sum_{j=2}^k w(x + q_1 - q_j) \right)^{p-1} - w^{p-1}(x) \right] \tilde{\varphi} \right| \\
& \leq C \left(\sum_{j=2}^k w(x + q_1 - q_j) \right)^{p-1} \leq C \left(\sum_{j=2}^k \frac{1}{|x + q_1 - q_j|^{N+2s}} \right)^{p-1} \\
& \leq C \left(\sum_{j=2}^k \frac{1}{|q_1 - q_j|^{N+2s}} \right)^{p-1} \leq C \left(\frac{k}{r} \right)^{(N+2s)(p-1)},
\end{aligned}$$

while for $p > 2$,

$$\begin{aligned}
& \left| \left[\left(w(x) + \sum_{j=2}^k w(x + q_1 - q_j) \right)^{p-1} - w^{p-1}(x) \right] \tilde{\varphi} \right| \\
& \leq C w^{p-2} \sum_{j=2}^k w(x + q_1 - q_j) + C \left(\sum_{j=2}^k w(x + q_1 - q_j) \right)^{p-1} \\
& \leq \sum_{j=2}^k \frac{C}{|x + q_1 - q_j|^{N+2s}} + C \left(\sum_{j=2}^k \frac{1}{|x + q_1 - q_j|^{N+2s}} \right)^{p-1} \\
& \leq \sum_{j=2}^k \frac{C}{|q_1 - q_j|^{N+2s}} + C \left(\sum_{j=2}^k \frac{1}{|q_1 - q_j|^{N+2s}} \right)^{p-1} \\
& \leq C \left(\frac{k}{r} \right)^{N+2s} + C \left(\frac{k}{r} \right)^{(N+2s)(p-1)} \leq C \left(\frac{k}{r} \right)^{N+2s}.
\end{aligned}$$

Hence $\tilde{g} \rightarrow 0$ uniformly on any compact set of \mathbb{R}^N as $k \rightarrow \infty$. Meanwhile, from

$$(-\Delta)^s \tilde{\varphi} + \tilde{\varphi} = (1 - V(x + q_1)) \tilde{\varphi} + p w^{p-1} \tilde{\varphi} + \tilde{g},$$

and Lemma 2.3, we obtain that

$$\sup_{x \neq y} \frac{|\tilde{\varphi}(x) - \tilde{\varphi}(y)|}{|x - y|^\beta} \leq C (\|(1 - V)\tilde{\varphi}\|_{L^\infty} + \|w^{p-1}\tilde{\varphi}\|_{L^\infty} + \|\tilde{g}\|_{L^\infty}) \leq C(\|\varphi\|_* + \|\tilde{g}\|_{L^\infty}) \leq C$$

where $\beta = \min\{1, 2s\}$. Hence up to a subsequence, we may assume that $\tilde{\varphi} \rightarrow \varphi_0$ uniformly on any compact set. It is easy to observe that φ_0 satisfies

$$\begin{cases} (-\Delta)^s \varphi_0 + \varphi_0 - p w^{p-1} \varphi_0 = 0 & \text{in } \mathbb{R}^N, \\ \varphi_0 \in H_s, \\ \int_{\mathbb{R}^N} \frac{\partial w}{\partial x^1} \varphi_0 = 0, \end{cases} \quad (4.12)$$

where $x = (x^1, \dots, x^N)$. Besides, we know, from Remark 2.2, that

$$\int_{B_R(0)} \varphi_0^2 \leq \int_{B_R(0)} \tilde{\varphi}_k^2 = \int_{B_R(q_1)} \varphi_k^2 \leq \|\varphi_k\|_*^2 \int_{B_R(q_1)} \rho^2 \leq C,$$

which means that $\varphi_0 \in L^2(\mathbb{R}^N)$. Then the non-degeneracy result in [20] implies that φ_0 must be a linear combination of the partial derivatives $\frac{\partial w}{\partial x^i}$, $i = 1, \dots, N$. But the symmetry and orthogonality condition yield that $\varphi_0 \equiv 0$, which is a contradiction to (4.9). The lemma is then proved. \square

5 The variational reduction and the proof of Theorem 1.2

In this section we first solve the intermediate nonlinear problem (3.3), i.e.

$$\begin{cases} (-\Delta)^s \varphi(x) + V(|x|)\varphi(x) - pW^{p-1}\varphi(x) = E + N(\varphi) + c \sum_{j=1}^k Z_j & \text{in } \mathbb{R}^N, \\ \varphi \in H_s, \\ \int_{\mathbb{R}^N} Z_j \varphi = 0 & \text{for any } j = 1, \dots, k. \end{cases}$$

Then we solve the final nonlinear problem (3.2) variationally.

Proposition 5.1. *Assume that k large enough, for any $r \in \left[\frac{1}{C_0} k^{\frac{N+2s}{N+2s-m}}, C_0 k^{\frac{N+2s}{N+2s-m}} \right]$, the problem (3.3) has a unique small solution $\varphi = \Phi(r)$ with*

$$\|\varphi\|_* \leq C \left(\frac{k}{r} \right)^{\min\{N+2s, (N+2s)p-\mu\}} + \frac{C}{r^{N+2s-\mu}} + \frac{C}{r^m} = o\left(\frac{1}{r^{m/2}} \right).$$

Furthermore, the map $r \rightarrow \Phi(r)$ is of class C^1 , and

$$\|\Phi'(r)\|_* \leq C \left(\frac{k}{r} \right)^{\min\{N+2s, (N+2s)p-\mu\}} + \frac{C}{r^{N+2s-\mu}} + \frac{C}{r^m}.$$

Proof. Problem (3.3) can be written as the fixed point problem

$$\varphi = T(E + N(\varphi)) =: \mathcal{A}(\varphi) \quad \text{for } \varphi \in H_s.$$

Let

$$\mathfrak{F} = \{\varphi \in H_s \mid \|\varphi\|_* \leq s_0\},$$

where $s_0 > 0$ is a small number determined later.

If $\varphi \in \mathfrak{F}$, either $1 < p \leq 2$,

$$\|N(\varphi)\|_* \leq C \|\varphi^p\|_* \leq C \|\varphi\|_*^p \|\rho\|_{L^\infty(\mathbb{R}^N)}^{p-1} \leq C \|\varphi\|_*^p;$$

or $p > 2$,

$$\|N(\varphi)\|_* \leq C \|\varphi^2 W^{p-2}\|_* + C \|\varphi^p\|_* \leq C \|\varphi\|_*^2 \|\rho W\|_{L^\infty(\mathbb{R}^N)} + C \|\varphi\|_*^p \|\rho\|_{L^\infty(\mathbb{R}^N)}^{p-1} \leq C \|\varphi\|_*^2.$$

By Proposition 4.1 and Lemma 3.1,

$$\|\mathcal{A}(\varphi)\|_* \leq C (\|E\|_* + \|N(\varphi)\|_*) \leq C \|E\|_* + C (\|\varphi\|_* + \|\varphi\|_*^{p-1}) \|\varphi\|_* \leq s_0$$

if we choose $C(s_0 + s_0^{p-1}) \leq \frac{1}{2}$ and k large enough such that

$$C \left(\frac{k}{r} \right)^{\min\{N+2s, (N+2s)p-\mu\}} + \frac{C}{r^{N+2s-\mu}} + \frac{C}{r^m} \leq \frac{1}{2} s_0.$$

On the other hand, for any $\varphi_i \in H_s$, $i = 1, 2$,

$$|N(\varphi_1) - N(\varphi_2)| = |N'(t)(\varphi_1 - \varphi_2)|$$

where t lies between φ_1 and φ_2 .

For $1 < p \leq 2$, $|N'(t)| \leq C|t|^{p-1} \leq C(|\varphi_1|^{p-1} + |\varphi_2|^{p-1})$ which tells us that

$$\begin{aligned} \|N(\varphi_1) - N(\varphi_2)\|_* &\leq C \|\varphi_1 - \varphi_2\|_* (\|\varphi_1\|_{L^\infty(\mathbb{R}^N)}^{p-1} + \|\varphi_2\|_{L^\infty(\mathbb{R}^N)}^{p-1}) \\ &\leq C \|\varphi_1 - \varphi_2\|_* (\|\varphi_1\|_*^{p-1} + \|\varphi_2\|_*^{p-1}) \|\rho\|_{L^\infty(\mathbb{R}^N)}^{p-1} \\ &\leq C \|\varphi_1 - \varphi_2\|_* (\|\varphi_1\|_*^{p-1} + \|\varphi_2\|_*^{p-1}) \\ &\leq C s_0^{p-1} \|\varphi_1 - \varphi_2\|_* \leq \frac{1}{2} \|\varphi_1 - \varphi_2\|_* \end{aligned}$$

provided s_0 small enough. And for $p > 2$, $|N'(t)| \leq C(W^{p-2}|t| + |t|^{p-1})$, from which we can deduce that

$$\begin{aligned} & \|N(\varphi_1) - N(\varphi_2)\|_* \\ & \leq C\|\varphi_1 - \varphi_2\|_* \left[\|\rho W\|_{L^\infty(\mathbb{R}^N)} (\|\varphi_1\|_* + \|\varphi_2\|_*) + (\|\varphi_1\|_*^{p-1} + \|\varphi_2\|_*^{p-1}) \|\rho\|_{L^\infty(\mathbb{R}^N)}^{p-1} \right] \\ & \leq C\|\varphi_1 - \varphi_2\|_* (\|\varphi_1\|_* + \|\varphi_2\|_* + \|\varphi_1\|_*^{p-1} + \|\varphi_2\|_*^{p-1}) \\ & \leq C(s_0 + s_0^{p-1})\|\varphi_1 - \varphi_2\|_* \leq \frac{1}{2}\|\varphi_1 - \varphi_2\|_* \end{aligned}$$

with s_0 small enough.

Thus we obtain that \mathcal{A} is a contraction mapping and the problem (3.3) has a unique solution φ . Obviously according to Lemma 3.1,

$$\|\varphi\|_* \leq C \left(\frac{k}{r} \right)^{\min\{N+2s, (N+2s)p-\mu\}} + \frac{C}{r^{N+2s-\mu}} + \frac{C}{r^m}.$$

For the proof of $\Phi(r) \in C^1$, please refer to [10]. Here we don't repeat it. \square

Next, we will use the above introduced ingredients to find existence results for the nonlinear problem (3.2), i.e. the equation

$$(-\Delta)^s u + V(x)u - u_+^p = 0. \quad (5.1)$$

Set the following energy functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} u(-\Delta)^s u + V(x)u^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} u_+^{p+1} \quad (5.2)$$

whose nontrivial critical points are solutions to (1.6).

We want to find a solution of (5.1) with the form $U = W + \varphi$ where $\varphi = \Phi(r)$ is found in Proposition 5.1. Then it is easy to observe that

$$(-\Delta)^s U + VU - U_+^p = c \sum_{j=1}^k Z_j.$$

Hence we need to find suitable r such that the coefficient $c = 0$. The problem can be formulated variationally as follows.

Lemma 5.1. *Let $F(r) = J(U) = J(W + \Phi(r))$, then $c = 0$ if and only if $F'(r) = 0$.*

Proof. Assume that \tilde{U} is the unique s -harmonic extension of $U = W + \Phi(r)$, then the well-known computation by Caffarelli and Silvestre [5] shows that

$$F(r) = \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} |\nabla \tilde{U}|^2 y^{1-2s} dx dy + \frac{1}{2} \int_{\mathbb{R}^N} V(|x|)U^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} U_+^{p+1}.$$

So with $\partial_r U = \partial_r W + \Phi'(r) = \sum_{i=1}^k Z_i + \Phi'(r)$, (4.3), (4.4) and Proposition 5.1,

$$\begin{aligned} F'(r) &= \int_{\mathbb{R}_+^{N+1}} \nabla \tilde{U} \cdot \nabla (\partial_r \tilde{U}) y^{1-2s} + \int_{\mathbb{R}^N} V(|x|)U \partial_r U - \int_{\mathbb{R}^N} U_+^p \partial_r U \\ &= \int_{\mathbb{R}^N} ((-\Delta)^s U + V(|x|)U - U_+^p) \partial_r U = c \sum_{j=1}^k \int_{\mathbb{R}^N} Z_j \partial_r U \\ &= c \sum_{i,j=1}^k \int_{\mathbb{R}^N} Z_j Z_i + c \sum_{j=1}^k \int_{\mathbb{R}^N} Z_j \Phi'(r) \\ &= c \left(\frac{k}{N} \int_{\mathbb{R}^N} (w'(|x|))^2 dx + kO\left(\left(\frac{k}{r}\right)^{N+2s}\right) + O(\|\Phi'(r)\|_* \sum_{j=1}^k \int_{\mathbb{R}^N} |Z_j| \rho) \right) \\ &= ck \left(\frac{1}{N} \int_{\mathbb{R}^N} (w'(|x|))^2 dx + o(1) \right), \end{aligned}$$

with k large enough. The proof is finished. \square

Now our task is to find a critical of the functional $F(r)$. We have the following expansion of $F(r)$.

Proposition 5.2. *There exists k_0 such that for any $k \geq k_0, r \in I_0$, the following expansion holds*

$$F(r) = k \left[A_1 + \frac{B_1}{r^m} - \frac{B_2 k^{N+2s}}{r^{N+2s}} + o(r^{-m}) \right] \quad (5.3)$$

where A_1, B_1, B_2 are universal positive constants defined in Proposition 6.1 and the interval I_0 is given by

$$I_0 = \left[\frac{1}{C_0} k^{\frac{N+2s}{N+2s-m}}, C_0 k^{\frac{N+2s}{N+2s-m}} \right].$$

Proof. Since $U = W + \varphi$, let us expand $J(U)$ at W and get that

$$\begin{aligned} J(U) &= \frac{1}{2} \int_{\mathbb{R}^N} U(-\Delta)^s U + VU^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} U_+^{p+1} \\ &= J(W) + \int_{\mathbb{R}^N} [(-\Delta)^s U + VU - U_+^p] \varphi - \frac{1}{2} \int_{\mathbb{R}^N} (\varphi(-\Delta)^s \varphi + V\varphi^2 - pW^{p-1}\varphi^2) \\ &\quad - \frac{1}{p+1} \int_{\mathbb{R}^N} \left((W + \varphi)_+^{p+1} - W^{p+1} - (p+1)W^p\varphi - \frac{p(p+1)}{2}W^{p-1}\varphi^2 \right) \\ &\quad + \int_{\mathbb{R}^N} (U_+^p - W^p - pW^{p-1}\varphi) \varphi. \end{aligned}$$

Since $\int_{\mathbb{R}^N} \varphi Z_j = 0$ for all $j = 1, \dots, k$, the second term disappears. From Remark 2.2, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} (\varphi(-\Delta)^s \varphi + V\varphi^2 - pW^{p-1}\varphi^2) \right| \\ &= \int_{\mathbb{R}^N} |E + N(\varphi)| |\varphi| \leq Ck (\|E\|_* + \|N(\varphi)\|_*) \|\varphi\|_* \int_{\Omega_1} \rho^2 \\ &\leq Ck (\|E\|_* + \|N(\varphi)\|_*) \|\varphi\|_* = ko(r^{-m}). \end{aligned}$$

Similarly, it is easy to see that

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \left((W + \varphi)_+^{p+1} - W^{p+1} - (p+1)W^p\varphi - \frac{p(p+1)}{2}W^{p-1}\varphi^2 \right) \right| \\ &\leq C \int_{\mathbb{R}^N} |\varphi|^{\min\{p+1, 3\}} \leq Ck \|\varphi\|_*^{\min\{p+1, 3\}} \int_{\Omega_1} \rho^2 = ko(r^{-m}), \end{aligned}$$

and

$$\left| \int_{\mathbb{R}^N} (U_+^p - W^p - pW^{p-1}\varphi) \varphi \right| = O \left(\int_{\mathbb{R}^N} W^{p-1}\varphi^2 \right) = kO(\|\varphi\|_*^2) = ko(r^{-m}).$$

The proof is completed. \square

Proof of Theorem 1.2. Now consider the following problem $\max_{r \in I_0} F(r)$. We want to verify that the maximum points lie in the interior of the interval I_0 . For this, let

$$r_0 = \left(\frac{(N+2s)B_2}{mB_1} \right)^{\frac{1}{N+2s-m}} k^{\frac{N+2s}{N+2s-m}} \in I_0$$

for large positive constant C_0 . It is easy to show that for large k ,

$$F(r_0) = kA_1 + k^{1-\frac{(N+2s)m}{N+2s-m}} B_1 \left(\frac{mB_1}{(N+2s)B_2} \right)^{\frac{m}{N+2s-m}} \frac{N+2s-m}{N+2s} + o \left(k^{1-\frac{(N+2s)m}{N+2s-m}} \right).$$

On the other hand, it always holds that

$$\begin{aligned} F \left(\frac{1}{C_0} k^{\frac{N+2s}{N+2s-m}} \right) &= kA_1 + k^{1-\frac{(N+2s)m}{N+2s-m}} (B_1 C_0^m - B_2 C_0^{N+2s}) + o \left(k^{1-\frac{(N+2s)m}{N+2s-m}} \right) \\ &< kA_1 + k^{1-\frac{(N+2s)m}{N+2s-m}} B_1 \left(\frac{mB_1}{(N+2s)B_2} \right)^{\frac{m}{N+2s-m}} \frac{N+2s-m}{2(N+2s)} \end{aligned}$$

and

$$\begin{aligned}
F(C_0 k^{\frac{N+2s}{N+2s-m}}) &= kA_1 + k^{1-\frac{(N+2s)m}{N+2s-m}} \left(\frac{B_1}{C_0^m} - \frac{B_2}{C_0^{N+2s}} \right) + o\left(k^{1-\frac{(N+2s)m}{N+2s-m}}\right) \\
&< kA_1 + k^{1-\frac{(N+2s)m}{N+2s-m}} \frac{B_1}{C_0^m} + o\left(k^{1-\frac{(N+2s)m}{N+2s-m}}\right) \\
&< kA_1 + k^{1-\frac{(N+2s)m}{N+2s-m}} B_1 \left(\frac{mB_1}{(N+2s)B_2} \right)^{\frac{m}{N+2s-m}} \frac{N+2s-m}{2(N+2s)},
\end{aligned}$$

if we choose C_0 large enough such that

$$B_1 C_0^m - B_2 C_0^{N+2s} < 0, \quad \frac{B_1}{C_0^m} < B_1 \left(\frac{mB_1}{(N+2s)B_2} \right)^{\frac{m}{N+2s-m}} \frac{N+2s-m}{4(N+2s)}$$

which can be done because of $0 < m < N+2s$. If we let $F(r_1) = \max_{r \in I_0} F(r)$, then r_1 is an interior point of I_0 and thus $F'(r_1) = 0$, which gives a critical point of $F(r)$.

Therefore Lemma 5.1 implies Theorem 1.2. \square

6 Appendix: Energy expansion

In this section, the important expansion of the energy at W is given. First we list the following lemmas, whose proofs can be found in [25].

Lemma 6.1. *For any $\alpha > 0$,*

$$\sum_{j=1}^k \frac{1}{(1+|x-x_j|)^\alpha} \leq C + C \sum_{j=2}^k \frac{1}{|x_1-x_j|^\alpha}, \quad \forall x \in \mathbb{R}^N$$

where $C > 0$ is a constant independent of k .

Lemma 6.2. *For any constant $0 < \sigma < N-2$, there is a constant $C > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}} dz \leq \frac{C}{(1+|y|)^\sigma}.$$

The proof of this lemma, a more general one actually, can also be found in [22, 29].

Lemma 6.3. *For any constant $0 \leq \sigma \leq \min\{\alpha, \beta\}$, there is a constant $C > 0$ such that, for any $i \neq j$,*

$$\frac{1}{(1+|x-q_i|)^\alpha} \frac{1}{(1+|x-q_j|)^\beta} \leq \frac{C}{|q_i-q_j|^\sigma} \left[\frac{1}{(1+|x-q_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1+|x-q_j|)^{\alpha+\beta-\sigma}} \right].$$

The proof of the above lemma may be found in [26].

Next we focus on the expansion of energy at W . Recall the positive least energy solution w to (1.9).

Proposition 6.1. *It holds that*

$$J(W) = k \left[A_1 + \frac{B_1}{r^m} - \frac{B_2 k^{N+2s}}{r^{N+2s}} + o\left(r^{-m} + \left(\frac{k}{r}\right)^{N+2s}\right) \right], \quad (6.1)$$

where

$$A_1 = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} w^{p+1}(x) dx, \quad B_1 = \frac{a}{2} \int_{\mathbb{R}^N} w^2(x) dx$$

and B_2 are all positive numbers.

Proof. Recall that

$$q_j = \left(r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, \mathbf{0} \right), \quad j = 1, \dots, k,$$

where $\mathbf{0}$ is the zero vector in \mathbb{R}^{N-2} , $r \in \left[\frac{1}{C_0} k^{\frac{N+2s}{N+2s-m}}, C_0 k^{\frac{N+2s}{N+2s-m}} \right]$ for a large positive constant C_0 . By direct calculus, we get that

$$|q_1 - q_j| = 2r \sin \frac{(j-1)\pi}{k}, \quad 0 < c' \leq \frac{\sin \frac{(j-1)\pi}{k}}{\frac{(j-1)\pi}{k}} \leq c'',$$

from which we can find that for any $\ell > 1$,

$$\sum_{j=2}^k \frac{1}{|q_j - q_1|^\ell} = \frac{1}{(2r)^\ell} \sum_{j=2}^k \frac{1}{(\sin \frac{(j-1)\pi}{k})^\ell} = C_\ell \left(\frac{k}{r}\right)^\ell + o\left(\left(\frac{k}{r}\right)^\ell\right), \quad (6.2)$$

where $C_\ell > 0$.

Denote

$$W_j(x) = w(x - q_j), \quad j = 1, \dots, k, \quad W(x) = \sum_{j=1}^k W_j(x).$$

$$\begin{aligned} J(W_1) &= \frac{1}{2} \int_{\mathbb{R}^N} [w(x - q_1)(-\Delta)^s w(x - q_1) + V(|x|)w^2(x - q_1)] dx - \frac{1}{p+1} \int_{\mathbb{R}^N} w^{p+1}(x - q_1) dx \\ &= J_1(w) + \frac{1}{2} \int_{\mathbb{R}^N} (V(|x|) - 1) w^2(x - q_1) dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} w^{p+1} dx + \frac{1}{2} \int_{\mathbb{R}^N} (V(|x - q_1|) - 1) w^2(x) dx. \end{aligned}$$

For any $\alpha > 0$, $x \in B_{r/2}(0)$, since

$$\frac{1}{|x - q_1|^\alpha} = \frac{1}{|q_1|^\alpha} \left[1 + O\left(\frac{|x|}{|q_1|}\right)\right], \quad (6.3)$$

we deduce that

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^N} (V(|x - q_1|) - 1) w^2(x) dx = \left(\int_{\{|x| < \frac{r}{2}\}} + \int_{\{|x| \geq \frac{r}{2}\}} \right) (V(|x - q_1|) - 1) w^2(x) dx \\ &= \frac{1}{2} \int_{\{|x| < \frac{r}{2}\}} \left[\frac{a}{|x - q_1|^m} + o\left(\frac{1}{|x - q_1|^m}\right) \right] w^2(x) dx + O\left(\int_{\{|x| \geq \frac{r}{2}\}} w^2(x) dx \right) \\ &= \frac{a}{2|q_1|^m} \int_{\{|x| < \frac{r}{2}\}} w^2(x) dx + O\left(r^{-(m+1)} \int_{\{|x| < \frac{r}{2}\}} |x| w^2(x) dx + r^{-(N+4s)} \right) + o(r^{-m}) \\ &= \frac{B_1}{r^m} + o(r^{-m}) + O\left(r^{-(N+4s)}\right) = \frac{B_1}{r^m} + o(r^{-m}), \end{aligned} \quad (6.4)$$

where the positive constant $B_1 = \frac{a}{2} \int_{\mathbb{R}^N} w^2(x) dx$. Recall that

$$\Omega_j = \left\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{y'}{|y'|}, \frac{q_j}{|q_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.$$

By symmetry, we can deduce that

$$\begin{aligned} J(W) &= \frac{1}{2} \int_{\mathbb{R}^N} W(-\Delta)^s W + V(|x|)W^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} W^{p+1} \\ &= \frac{1}{2} \int_{\mathbb{R}^N} W((-\Delta)^s W + W) + \frac{1}{2} \int_{\mathbb{R}^N} (V(|x|) - 1) W^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} W^{p+1} \\ &= \frac{1}{2} \int_{\mathbb{R}^N} W \sum_{j=1}^k W_j^p + \frac{k}{2} \int_{\Omega_1} (V(|x|) - 1) W^2 - \frac{k}{p+1} \int_{\Omega_1} W^{p+1} \\ &= \frac{k}{2} \int_{\mathbb{R}^N} w^{p+1} + \frac{k}{2} \sum_{j=2}^k \int_{\mathbb{R}^N} W_1^p W_j + \frac{k}{2} \int_{\Omega_1} (V(|x|) - 1) W^2 - \frac{k}{p+1} \int_{\Omega_1} W^{p+1}. \end{aligned}$$

Now let us do the computations term by term.

With (6.3) and (1.11) at hand, we find that

$$\sum_{j=2}^k \int_{\mathbb{R}^N} W_1^p W_j = \sum_{j=2}^k \int_{\mathbb{R}^N} w^p(|x - q_1|) w(|x - q_j|) dx$$

$$\begin{aligned}
&= \sum_{j=2}^k \int_{\mathbb{R}^N} w^p(|x|) w(|x + q_1 - q_j|) dx \\
&= \sum_{j=2}^k \int_{\{x \mid |x| \leq |q_1 - q_j|/2\}} w^p(|x|) \left[\frac{A}{|x + q_1 - q_j|^{N+2s}} + o\left(\frac{1}{|x + q_1 - q_j|^{N+2s}}\right) \right] dx \\
&\quad + \sum_{j=2}^k \int_{\{x \mid |x| \geq |q_1 - q_j|/2\}} O\left(\frac{1}{|q_1 - q_j|^{(N+2s)p}}\right) w(|x + q_1 - q_j|) dx \\
&= \sum_{j=2}^k \frac{A}{|q_1 - q_j|^{N+2s}} \int_{\mathbb{R}^N} w^p(x) dx + o\left(\sum_{j=2}^k \frac{1}{|q_1 - q_j|^{N+2s}}\right) + O\left(\sum_{j=2}^k \frac{1}{|q_1 - q_j|^{(N+2s)p}}\right) \\
&= \sum_{j=2}^k \frac{\tilde{B}_2}{|q_1 - q_j|^{N+2s}} + o\left(\left(\frac{k}{r}\right)^{N+2s}\right) \tag{6.5}
\end{aligned}$$

where the positive constant $\tilde{B}_2 = A \int_{\mathbb{R}^N} w^p$.

For any $x \in \Omega_1$, it is obvious that $|x - q_j| \geq |x - q_1|$ and $|x - q_j| \geq |q_j - q_1|/2$ for $j = 2, \dots, k$. Then for any $0 \leq \alpha \leq N + 2s$,

$$W_j(x) \leq \frac{C}{(1 + |x - q_j|)^{N+2s}} \leq \frac{C}{(1 + |x - q_1|)^\alpha |q_1 - q_j|^{N+2s-\alpha}}.$$

Hence for any $0 \leq \alpha < N + 2s - 1$,

$$\sum_{j=2}^k W_j(x) = O\left(\frac{1}{(1 + |x - q_1|)^\alpha} \sum_{j=2}^k \frac{1}{|q_1 - q_j|^{N+2s-\alpha}}\right) = O\left(\frac{1}{(1 + |x - q_1|)^\alpha} \left(\frac{k}{r}\right)^{N+2s-\alpha}\right). \tag{6.6}$$

Now we can deduce that

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega_1} (V(|x|) - 1) W^2 = \frac{1}{2} \int_{\Omega_1} (V(|x|) - 1) \left(W_1 + \sum_{j=2}^k W_j\right)^2 \\
&= \frac{1}{2} \int_{\Omega_1} (V(|x|) - 1) W_1^2 + O\left(\int_{\Omega_1} |V(|x|) - 1| W_1 \sum_{j=2}^k W_j + \left(\frac{k}{r}\right)^{2N+4s-2\alpha} \int_{\Omega_1} \frac{1}{(1 + |x - q_1|)^{2\alpha}}\right) \\
&= \frac{B_1}{r^m} + o(r^{-m}) + O\left(\left(\frac{k}{r}\right)^{N+2s} \int_{\Omega_1} |V(|x|) - 1| W_1 + \left(\frac{k}{r}\right)^{N+3s}\right) \\
&= \frac{B_1}{r^m} + o\left(r^{-m} + \left(\frac{k}{r}\right)^{N+2s}\right) + \left(\frac{k}{r}\right)^{N+2s} O\left(\int_{\Omega_1} |V(|x|) - 1| w(x - q_1)\right) \\
&= \frac{B_1}{r^m} + o\left(r^{-m} + \left(\frac{k}{r}\right)^{N+2s}\right) + \left(\frac{k}{r}\right)^{N+2s} O\left(\left(\int_{\{x \mid |x| \leq \frac{r}{2}\}} + \int_{\{x \mid |x| \geq \frac{r}{2}\}}\right) |V(x) - 1| w(x - q_1)\right) \\
&= \frac{B_1}{r^m} + o\left(r^{-m} + \left(\frac{k}{r}\right)^{N+2s}\right) + \left(\frac{k}{r}\right)^{N+2s} O\left(\frac{1}{r^s} + \frac{1}{r^m}\right) \\
&= \frac{B_1}{r^m} + o\left(r^{-m} + \left(\frac{k}{r}\right)^{N+2s}\right), \tag{6.7}
\end{aligned}$$

where we choose $\alpha = \frac{N+s}{2}$.

For the last term in the energy $J(W)$, it is not difficult to check that

$$\begin{aligned}
\frac{1}{p+1} \int_{\Omega_1} W^{p+1} &= \frac{1}{p+1} \int_{\Omega_1} \left(W_1 + \sum_{j=2}^k W_j \right)^{p+1} \\
&= \frac{1}{p+1} \int_{\Omega_1} W_1^{p+1} + \int_{\Omega_1} W_1^p \sum_{j=2}^k W_j + O \left(\int_{\Omega_1} W_1^{p-1} \left(\sum_{j=2}^k W_j \right)^2 \right) + O \left(\int_{\Omega_1} \left(\sum_{j=2}^k W_j \right)^{p+1} \right) \\
&= \frac{1}{p+1} \int_{\Omega_1} W_1^{p+1} + \int_{\Omega_1} W_1^p \sum_{j=2}^k W_j + O \left(\left(\sum_{j=2}^k \frac{1}{|q_j - q_1|^{\frac{N}{2} + 2s}} \right)^2 \right) \\
&\quad + O \left(\left(\sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s - \frac{N+(p-1)s}{p+1}}} \right)^{p+1} \right) \\
&= \frac{1}{p+1} \int_{\mathbb{R}^N} w^{p+1} + \sum_{j=2}^k \frac{\tilde{B}_2}{|q_1 - q_j|^{N+2s}} + O \left(\left(\frac{k}{r} \right)^{N+4s} \right).
\end{aligned} \tag{6.8}$$

Combining (6.5), (6.7) and (6.8), we get the desired expansion of energy

$$J(W) = k \left[A_1 + \frac{B_1}{r^m} - \frac{1}{2} \sum_{j=2}^k \frac{\tilde{B}_2}{|q_1 - q_j|^{N+2s}} + o \left(r^{-m} + \left(\frac{k}{r} \right)^{N+2s} \right) \right], \tag{6.9}$$

where

$$A_1 = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} w^{p+1}(x) dx, \quad B_1 = \frac{a}{2} \int_{\mathbb{R}^N} w^2(x) dx, \quad \tilde{B}_2 = A \int_{\mathbb{R}^N} w^p.$$

With (6.2) at hand, we finished the proof. \square

Acknowledgement. Wang is supported by NSFC (Project11371254). Zhao is supported by NSFC (Project 11101155) and by the Fundamental Research Funds for the Central Universities.

References

- [1] A. Ambrosetti, M. Badiale and S. Cingolani, Semiclassical states of nonlinear Schrödinger equations, Arch. Rational Mech. Anal. 140 (1997), 285-300.
- [2] A. Ambrosetti, A. Malchiodi and W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. I, Comm. Math. Phys. 235 (2003), 427-466.
- [3] A. Ambrosetti, A. Malchiodi and W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. II, Indiana Univ. Math. J. 53 (2004), 297-329.
- [4] C.J. Amick and J. Toland, Uniqueness and related analytic properties for the Benjamin-Ono equation-a nonlinear Neumann problem in the plane, Acta Math. 167 (1991), 107-126.
- [5] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007), no.7-9, 1245-1260.
- [6] D.M. Cao, E.S. Noussair and S. Yan, Existence and uniqueness results on single-peaked solutions of a semilinear problem, Ann. Inst. H. Poincaré Anal. NonLineaire 15 (1998), 73-111.
- [7] D.M. Cao, E.S. Noussair and S. Yan, Solutions with multiple peaks for nonlinear elliptic equations, Proc. Roy. Soc. Edinburgh Sect. A 129(1999), no.2, 235-264.
- [8] E.N. Dancer, K.Y. Lam and S. Yan, The effect of the graph topology on the existence of multi-peak solutions for nonlinear Schrödinger equations, Abstr. Appl. Anal. 3 (1998), 293-318.

- [9] E.N. Dancer and S. Yan, On the existence of multi-peak solutions for nonlinear field equations on \mathbb{R}^N , *Discrete Contin. Dynam. Systems* 6 (2000), 39-50.
- [10] Juan Dávila, M. del Pino and J. Wei, Concentrating standing waves for the fractional nonlinear Schrödinger equation, *J. Diff. Eqns.* 256 (2014), no.2, 858-892.
- [11] M. del Pino and P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differential Equations.* 4 (1996), 121-137.
- [12] M. del Pino and P. Felmer, Multi-peak bound states of nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré Anal. NonLineaire* 15 (1998), 127-149.
- [13] M. del Pino and P. Felmer, Semi-classical states for nonlinear Schrödinger equations, *J. Funct. Anal.* 149 (1997), 245-265.
- [14] M. del Pino and P. Felmer, Semi-classical states for nonlinear Schrödinger equations: a variational reduction method, *Math. Ann.* 324 (2002), 1-32.
- [15] M. del Pino, J.C. Wei and W. Yao, Intermediate reduction methods and infinitely many positive solutions of nonlinear Schrödinger equations with non-symmetric potentials, submit.
- [16] A. Felmer and S. Martinez, Thick clusters for the radially symmetric nonlinear Schrödinger equation, *Cal. Var. PDE* 31 (2008), 231-261.
- [17] A. Floer and A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.*, 69 (1986), no.3, 397-408.
- [18] P. Felmer, A. Quaas and J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* 142 (2012), no.6, 1237-1262.
- [19] R. Frank and E. Lenzmann, Uniqueness of non-linear ground states for fractional Laplacians in \mathbb{R} . *Acta Math.* 210 (2013), no.2, 261-318.
- [20] R. Frank, E. Lenzmann and L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, arxiv:1302.2652v1.
- [21] X.S. Kang and J.C. Wei, On interacting bumps of semi-classical states of nonlinear Schrödinger equations, *Adv. Diff. Eqn.* 5 (2000), 899-928.
- [22] Y. Li and W.-M. Ni, On conformal scalar curvature equations in \mathbb{R}^n . *Duke Math. J.* 57 (1988), no.3, 895-924.
- [23] E.S. Noussair and S.S. Yan, On positive multi-peak solutions of a nonlinear elliptic problem, *J. London Math. Soc.* 62 (2000), 213-277.
- [24] Y.J. Oh, On positive multi-lump bound states nonlinear Schrödinger equations under multiple well potential, *Comm. Math. Phys.* 131 (1990), 223-253.
- [25] L.P. Wang, J.C. Wei and S.S. Yan, A Neumann problem with critical exponent in non-convex domains and Lin-Ni's conjecture, *Tran. American Math. Society* 362(2010), no.9, 4581-4615.
- [26] L.P. Wang and C.Y. Zhao, Infinitely many solutions for the prescribed boundary mean curvature problem in \mathbb{B}^N , *Canad. J. Math.* 65 (2013), no.4, 927-960.
- [27] X. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, *Comm. Math. Phys.* 153 (1993), 229-243.
- [28] J.C. Wei and S.S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger equations in \mathbb{R}^N , *Calc. Var. PDE*, 37 (2010), no.3-4, 423-439.
- [29] J.C. Wei and C.Y. Zhao, Non-compactness of the prescribed Q-curvature problem in large dimensions, *Calculus of Variations and Partial Differential Equations* 46 (2013), 123-164.
- [30] C.Y. Zhao, On the number of interior peaks of solutions to a non-autonomous singularly perturbed Neumann problem, *Proc. Roy. Soc. Edinburgh Sect. A* 139 (2009), no. 2, 427-448.